The flow past a rapidly rotating circular cylinder in a uniform stream

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SUMMARY

A circular cylinder is rotating in a uniform stream of viscous liquid, the circumferential velocity of the cylinder being much greater than that of the uniform stream. It is shown that the effect of the uniform stream can be regarded as a small perturbation of the rotary flow due to the cylinder, and that a uniformly valid first approximation to the flow field can be obtained in this way. This approximate solution has the interesting property that the circulation about any circular contour concentric with the cylinder is the same. The lift on the cylinder is shown to be that given by the classical formula, and the drag force is zero to the order of the approximation considered.

1. INTRODUCTION

In inviscid fluid theory an important type of two-dimensional flow is that generated when an infinite cylinder is placed in a uniform stream, with its generators normal to the plane of flow. Since the space outside the cylinder is doubly connected the velocity potential is determined only if the circulation $K = \int_\Gamma \mathbf{u} \, . \, d\mathbf{l}$ is specified about some contour Γ encircling the cylinder. The cylinder is then found to experience a force of magnitude $K\rho U$ per unit length perpendicular to the direction of the uniform stream, where ρ is the fluid density, and U is the velocity of the uniform stream. This result is due to Kutta (1910). Taylor (1925) showed that this same formula for the force on the cylinder is valid for a viscous fluid, provided that K is defined by a contour which is everywhere distant from the cylinder, and which crosses the cylinder's wake normally. This result explained the experimental work of Bryant & Williams (1925), who had compared the measured value of the lift with the value obtained from the inviscid formula by inserting the measured value of K.

To determine the value of K for given boundary conditions on the cylinder it is necessary to take the viscosity of the fluid into account. Sometimes this can be done without reference to the detailed viscous flow pattern. In the case of a slender aerofoil terminating in a cusp, for instance, the well-known Kutta–Joukowski condition, that K must be chosen so that the velocity of the potential flow is finite at the cusp, is a very plausible version of the requirement that steady viscous flow should be possible in the boundary layers near the cusp. In general, however, it is not possible to determine K without solving the Navier–Stokes equations for the flow, and it is with one such case that the present paper is concerned.

The problem considered is that of a circular cylinder of radius a rotating with a circumferential velocity V in a uniform stream of velocity U. The viscous stresses set up by the cylinder's rotation induce a circulation in the fluid, and the cylinder experiences a transverse force in accordance with the general theory mentioned above. The existence of this force was discovered by Magnus in 1852, and the phenomenon is now usually known as the 'Magnus Effect'.

When the Reynolds number Va/ν is large the viscous forces may be expected to be small outside the boundary layer, and therefore the well-known inviscid theory will apply outside the boundary layer. The flow pattern given by the inviscid theory depends critically on the ratio $2\pi Ua/K$. If this ratio exceeds $\frac{1}{2}$ there are two stagnation points on the circumference of the cylinder. If $2\pi Ua/K < \frac{1}{2}$, on the other hand, there are no stagnation points on the cylinder, and there is a region of closed streamlines near the cylinder. In this case, closed boundary layers of a well-behaved type might be expected to form if the Reynolds number of the flow were large. If the circulation is produced solely by the cylinder's rotation it is plausible to assume that K is $O(aV)$, so that if U/V is small closed boundary layers may be expected to be formed. These are observed experimentally when $U/V < \frac{1}{4}$ (Prandtl & Tietjens 1936, plate 7). Wood (1957) has considered this case from a boundary layer point of view, by expanding the velocity in the boundary layer in powers of U/V and substituting this expansion into the boundary layer equations.

In the case $U/V = 0$ the problem has a simple exact solution for all values of R, and it is reasonable to expect that when $U/V \ll 1$ a solution might be obtained, for all values of R, by regarding the uniform stream as producing a small perturbation of this exact solution. It is the purpose of the present paper to examine this approximation. Since the perturbation velocity becomes comparable with the velocity of the basic flow at a great distance from the cylinder, the perturbation method is obviously in danger of being invalid in this region of the flow. In § 2 it is shown that this difficulty can be overcome by using two approximations, one appropriate to the region near the cylinder, and one appropriate to large distances from the cylinder. In particular a uniformly valid first approximation to the velocity distribution can be found for all values of the Reynolds number, and a uniformly valid second approximation to the velocity distribution is found when the Reynolds number is greater than $\sqrt{48}$. Further approximations are not obtained explicitly, but their general nature can be inferred, and their relation to Wood's expansion elucidated.

The remainder of the paper is concerned mainly with the first approximation; the nature of the flow pattern, and the forces on the cylinder, being considered in § 3 and § 4, respectively. It appears that, to the first order in U/V the circulation round every *circular* contour concentric with the cylinder is the same. Thus, to this order, either the circulation at the cylinder's circumference, or round a contour very distant from the cylinder can be inserted in the formula for the lift. This is verified in § 4 by a direct calculation of the lift. The drag force is also calculated, and is shown to be zero to this order in U/V. The results provided by a second approximation are also considered briefly in § 3 and § 4. In particular, the circulation round a distant contour is determined, and, in the limit of infinite Reynolds numbers, is shown to be the same as that predicted by Wood from boundary layer considerations.

2. The first approximation

Using polar coordinates (ar, θ) with the origin at the centre of the cylinder, and the line $\theta = 0$ in the direction of the uniform stream, the equation for the stream function $Va\psi$ is

$$\nabla^4\psi = R\left[\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}(\nabla^2\psi) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}(\nabla^2\psi)\right], \tag{1}$$

where, it will be remembered, $R = Va/\nu$ is the Reynolds number based on the circumferential speed of the cylinder. The boundary conditions to be satisfied by ψ are,

$$\psi = 0, \qquad \frac{\partial\psi}{\partial r} = -1, \qquad \text{at } r = 1, \tag{2}$$

$$\psi \sim \epsilon r \sin\theta \quad \text{as } r \to \infty, \tag{3}$$

where $\epsilon = U/V$ is the small expansion parameter.

When $\epsilon = 0$, the relevant exact solution of (1) is the irrotational purely rotary flow

$$\psi_0 = -\log r. \tag{4}$$

Here, in accordance with the ideas discussed in § 1, an approximate solution of the form

$$\psi = \psi_0 + \epsilon\psi_1 \tag{5}$$

is sought. The function ψ_1 must satisfy the boundary conditions

$$\psi_1 = \frac{\partial\psi_1}{\partial r} = 0 \quad \text{at } r = 1, \tag{6}$$

$$\psi_1 \sim r\sin\theta \quad \text{as } r \to \infty. \tag{7}$$

If terms in ϵ^2 are neglected, substitution of (5) in the exact flow equation (1) shows that ψ_1 satisfies the equation

$$\nabla^4\psi_1 - \frac{R}{r^2}\frac{\partial}{\partial\theta}(\nabla^2\psi_1) = 0. \tag{8}$$

The general solution of (8) can be obtained by assuming that it may be represented by a Fourier series

$$\psi_1 = \sum_{-\infty}^{+\infty} f_n(r)e^{ni\theta}; \tag{9}$$

this is suggested by the fact that ψ_1 must be a single-valued function of θ. Fortunately, it is found that the boundary conditions (2) and (3) can be satisfied by taking only the terms for $n = \pm 1$, so that convergence questions do not arise. It is then easily found that

$$\psi_1 = \mathscr{R}\{e^{i\theta}[-i(r-r^{-1}) - Fr^{-1} + Fr^{(2-p+iq)}]\}. \tag{10}$$

In this solution

$$F = \frac{q+(3-p)i}{q^2+(3-p)^2} \tag{11}$$

and p and q are real constants given by

$$p, -q = [\tfrac{1}{2}\{\sqrt{(1+R^2)} \pm 1\}]^{1/2}. \tag{12}$$

A description of the flow pattern represented by (10) will be given in § 3; the remainder of the present section is concerned with the validity of the approximations leading to this solution.

The exact solution of (1), (2) and (3) is, of course, unknown, so that the validity of (10) must be considered by indirect methods. Thus if (10) is substituted into (1), the value of the terms that are neglected when (8) is derived can be calculated; and the magnitude of these neglected terms compared with the magnitude of the retained terms determines whether or not the approximation is consistent. It must be emphasized that this procedure cannot rigorously establish the validity of the approximation; for if the approximation is invalid it cannot be used to calculate the neglected terms, and the procedure fails. However, since the approximation is physically plausible, it seems reasonable to accept it as valid if the procedure should show it to be consistent.

The exact equation for ψ_1 is

$$\nabla^4\psi_1 - \frac{R}{r^2}\frac{\partial}{\partial\theta}(\nabla^2\psi_1) = R\epsilon\left[\frac{\partial\psi_1}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r}(\nabla^2\psi_1) - \frac{\partial\psi_1}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi_1)\right], \tag{13}$$

where the second term on the left-hand side is the retained inertia term, and the right-hand side consists of the previously neglected inertia terms. When $r = O(1)$ the retained inertia term is obviously dominant. At large values of r, however, one finds, on inserting (10) into this equation, that the retained inertia term is $O(r^{-p-2})$ whilst the largest neglected inertia term is $O(\epsilon r^{-p-1})$. Thus the approximation breaks down when $r = O(1/\epsilon)$, and since the outer boundary condition (7) must be applied, the validity of the solution (10) requires further justification, even when $r \ll 1/\epsilon$.

Now when $r = O(1/\epsilon)$ the contributions to the velocity from both the basic rotary flow and the uniform stream are $O(\epsilon)$, and it is reasonable to suppose that the exact solution has this same property. If, then, the equation (8) is solved subject to the exact boundary conditions on the cylinder and the order of magnitude conditions on the velocity when $r = O(1/\epsilon)$ one finds that the solution must be of the form $C\psi_1$ where C is a constant of order unity and ψ_1 is given by (10). In order to proceed any further the nature of the flow in the region $O(1) < \epsilon r < \infty$ must be examined. Now the rotational part of the velocity distribution corresponding to this solution is $O(\epsilon^p)$ when $r = O(1/\epsilon)$ where, from (12), p is always greater than unity and increases with R. Hence, with the mild assumption that the order of magnitude of the vorticity does not increase with r in the outer region, the flow in this region is, to a first approximation, irrotational. The problem in the outer region is therefore that of finding an irrotational stream function which satisfies the boundary conditions

$$\psi \sim \rho \sin\theta \quad \text{as } \rho = \epsilon r \to \infty, \tag{14}$$

$$\psi \sim C(\log\epsilon - \log\rho + \rho\sin\theta + o(\rho)) \quad \text{as } \rho \to 0. \tag{15}$$

Equation (15) is the condition that the inner and outer solutions should match. Clearly a solution is possible only if $C = 1$, and the unique outer solution is then

$$\psi = -\log r + \epsilon r \sin\theta. \tag{16}$$

Thus, subject to the assumed conditions on the orders of magnitude of the velocity and vorticity distributions, it has been established that (10) provides a first approximation to the velocity distribution throughout the whole flow field. It should be noted, however, that this solution provides an approximation to the vorticity distribution only when $r < O(1/\epsilon)$.

It is clear that, in general, one cannot obtain a better approximation to ψ without taking account of viscous effects in the outer region. A method of obtaining outer expansions which included the effects of viscosity was considered recently by Proudman & Pearson (1957), and it is reasonable to suppose that their technique could be applied to this case*. However, when $p > 2$, the discussion of the rotational term given above shows that it is $O(\epsilon^2)$ in the outer region, so that the flow in the outer region is irrotational to a second approximation. In this case one seeks a second approximation in the inner region by writing

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2, \tag{17}$$

and neglecting terms in ϵ^3 when (17) is substituted into (1). One finds that ψ_2 must satisfy the equation

$$\nabla^4\psi_2 - \frac{R}{r^2}\frac{\partial}{\partial\theta}(\nabla^2\psi_2) = R\left[\frac{\partial\psi_1}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r}(\nabla^2\psi_1) - \frac{\partial\psi_1}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi_1)\right]. \tag{18}$$

The general solution of (18) can be effected by expanding ψ_2 as a complex Fourier series. One then imposes the exact boundary conditions

$$\psi_2 = \frac{\partial\psi_2}{\partial r} = 0, \quad \text{at } r = 1, \tag{19}$$

and the order of magnitude requirement that the vorticity corresponding to this solution should be $O(\epsilon^3)$ when $r = O(1/\epsilon)$. These conditions show that the most general form ψ_2 can have is

$$\psi_2 = \lambda\psi_1 - f'(1)\log r - f(1) + f(r) + \mathscr{R}\{e^{2i\theta}[\alpha r^2 + \beta r^{-2} + \gamma r^\delta + g(r)]\}. \tag{20}$$

In (20), $f(r)$ and $g(r)$ are known functions arising from the inhomogeneous

* Filon (1928) considered some general properties of the flow at great distances for a cylinder when circulation is present, but did not consider the possibility of matching his solutions to an inner expansion.

part of (18), and α, β, λ, and γ are unknown constants satisfying the equations

$$\left.\begin{aligned} \alpha+\beta+\gamma+g(1) &= 0, \\ 2\alpha-2\beta+\delta\lambda+g'(1) &= 0. \end{aligned}\right\} \tag{21}$$

The constant δ is the value of $2+(4+2iR)^{1/2}$ whose real part is negative. The most general form that the expansion in the irrotational outer region can assume is

$$\psi = \log\epsilon+(-\log\rho+\rho\sin\theta)+\epsilon\mathscr{R}[A(\log\epsilon-\log\rho)+Be^{i\theta}/\rho+Ce^{2i\theta}/\rho^2+\ldots]+$$
$$+\epsilon^2\mathscr{R}[A'(\log\epsilon-\log\rho)+B'e^{i\theta}/\rho+C'e^{2i\theta}/\rho^2+\ldots]. \tag{22}$$

If one puts $r = \rho/\epsilon$ in (20) and imposes the condition that the expansions (17) and (22) should match when $\rho \to 0$, one finds that

$$\left.\begin{aligned} \lambda = 0, \qquad \alpha &= 0, \\ A = B = C = \ldots &= 0, \\ A' = f'(1), \qquad B' = C' = \ldots &= 0. \end{aligned}\right\} \tag{23}$$

Thus ψ_2 is uniquely determined, and the outer expansion is now

$$\psi = \log\epsilon+(-\log\rho+\rho\sin\theta)+\epsilon^2 f'(1)(\log\epsilon-\log\rho). \tag{24}$$

The calculation of the functions $f(r)$ and $g(r)$ is straightforward, but rather lengthy, and details will not be given.

The solution just obtained is valid only when $p > 2$, or from (12), $R > \sqrt{48}$. The determination of the second approximation would, as has been remarked, require the consideration of a viscous outer expansion if this condition were not satisfied, but in view of the greater practical interest in higher Reynolds numbers this is not attempted.

As R is still further increased the vorticity in the outer region decreases in order of magnitude, and one can find progressively more terms of an inner expansion in powers of ϵ. Eventually in the limit of infinite Reynolds number, one would arrive at the situation considered by Wood (1957) in which one expands the velocity in powers of ϵ and determines the coefficients by substitution in the boundary layer equations. It is not clear that in the case of finite Reynolds numbers the inner expansion will be a power series in ϵ, as it may well contain terms of the type $\epsilon^n(\log\epsilon)^m$.

3. The nature of the flow pattern

The flow field surrounding the rotating cylinder presents some interesting features, and it will be discussed in this section on the basis of the approximations obtained in § 2.

The first step in elucidating the flow pattern is to calculate the circulation about a circular contour of radius r, concentric with the cylinder. This is given by the formula

$$K(r) = -Var\int_0^{2\pi}\frac{\partial}{\partial r}(\psi_0+\epsilon\psi_1)\,d\theta, \tag{25}$$

and on inserting the expressions given in § 2 for the functions ψ_0 and ψ_1 one finds that ψ_1 makes no contribution to the integral, so that

$$K(r) = 2\pi Va. \tag{26}$$

Thus the circulation about all circular contours concentric with the cylinder is the same, and is equal to the circulation at the circumference of the cylinder. By Stokes's theorem, this implies that the net amount of vorticity contained between any two such circles is zero. It is interesting to recall that Taylor (1925) made the hypothesis that the net amount of vorticity in the wake contained between lines perpendicular to the wake was zero.

If $p > 2$, the second approximation derived in § 2 can be used to calculate the circulation. It is found that in the inner region

$$K(r) = 2\pi Va[1+\epsilon^2 f'(1) - rf'(r)] \tag{27}$$

and that in the outer region

$$K(r) = 2\pi Va[1+\epsilon^2 f'(1)]. \tag{28}$$

The function $f(r)$ is given by

$$f(r) = R\mathscr{R}\left[\frac{2ir^{\sigma+\bar\sigma}}{(\sigma+\bar\sigma)^2(\sigma+\bar\sigma-2)(\sigma+1)} - \frac{ir^{\bar\sigma+1}}{(\bar\sigma+1)^2} + \frac{ir^{\bar\sigma+1}(\sigma-1)}{(\bar\sigma-1)(\bar\sigma-3)(\sigma+1)}\right], \tag{29}$$

where $\sigma = 2-p+iq$, and the overbar denotes the complex conjugate. Thus the circulation at very large distances is constant, and is given by

$$K = 2\pi Va\left[1+\frac{\epsilon^2 q(3p-5)}{2(p^2-1)(p^2-3p+4)}\right]. \tag{30}$$

It may be shown from (12) that

$$p,\ -q \sim (\tfrac{1}{2}R)^{1/2} \quad \text{as } R\to\infty, \tag{31}$$

and so one finds that

$$K \to 2\pi Va(1-3\epsilon^2) \quad \text{as } R\to\infty. \tag{32}$$

The result (32) was obtained by Wood (1957) from boundary layer considerations.

A first approximation to the distribution of the vorticity ζ can be calculated from (10):

$$\zeta = -\epsilon Va^{-1}\mathscr{R}\{r^{-p+iq}e^{i\theta}F[(2-p+iq)^2-1]\}. \tag{33}$$

The result assumes a simpler form when R is large compared with unity. It is then found that

$$\zeta \sim Ua^{-1}R^{1/2}r^{-\sqrt{(\frac{1}{2}R)}}\cos(\tfrac{3}{4}\pi+\theta-(\tfrac{1}{2}R)^{1/2}\log r). \tag{34}$$

This expression shows that the vorticity suffers a rapid oscillatory decay as the wall of the cylinder is left, and that it is effectively confined to a boundary layer of thickness $O(aR^{-1/2})$. Wood (1957) has shown that this oscillatory character is an inherent property of closed boundary layers.

A closer examination of (10) throws further light on the nature of the flow pattern. The first term is simply the potential flow around the cylinder induced by the uniform stream. The second and third terms arise from the viscosity of the fluid. At large Reynolds numbers, the third term decays rapidly as r increases, and corresponds to a thin, closed, boundary layer attached to the cylinder. The second term, on the other hand, decays only as fast as r^{-1}, and corresponds to an irrotational secondary flow induced by the normal velocities at the outer edge of the boundary layer. The character of this secondary flow is made clear by writing down the stream function from which it can be derived. When R is large compared with unity this stream function is

$$\psi' \sim Uar^{-1}R^{-1/2}\cos(\theta + \tfrac{1}{4}\pi). \tag{35}$$

This secondary flow, like all such flows, is irrotational, and decreases in magnitude as $R \to \infty$. The inflow into the boundary layer is shown diagrammatically in figure 1.

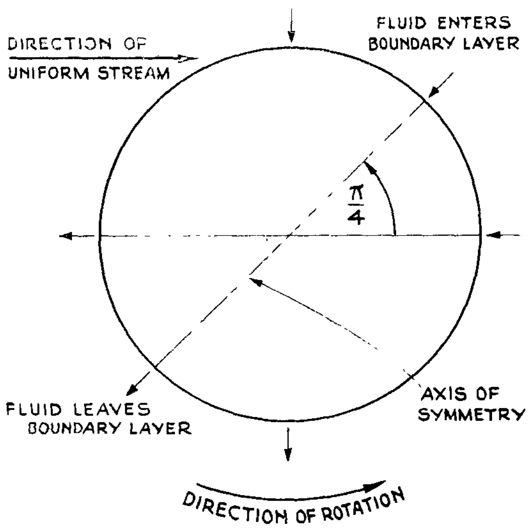

Figure 1. The distribution of inflow velocity into the boundary layer.

The mechanism producing the secondary flow is of the same general type as is found in other rotary flows. The viscous forces modify the velocity of a given fluid particle, so that the centrifugal force due to the curvature of the streamlines no longer balances the normal pressure gradient. There seems to be no obvious physical reason for the angular distribution of the flow, particularly the symmetry about the line $\theta = \frac{1}{4}\pi$.

It appears, then, that at large Reynolds numbers the flow pattern outside the boundary layer is very nearly that predicted by the inviscid theory. More precisely, at any point outside the boundary layer,

$$\text{(viscous stream function)} = \text{(inviscid stream function)} + O(R^{-1/2}). \tag{36}$$

In particular, there is no wake; a fact which will be found to have important consequences for the drag force experienced by the cylinder.

The formation of a boundary layer as $R \to \infty$, with the corresponding occurrence of large velocity gradients, may make some of the terms neglected in the derivation of (8) of comparable order to those retained. Inspection of the exact equation (13) for ψ_1 reveals that the terms retained are $O(\psi\epsilon/\delta^4)$ whilst the largest neglected terms are $O(\psi^2\epsilon^2 R/\delta^3)$, where δ is the dimensionless boundary layer thickness. In the boundary layer $\psi = O(R^{-1/2})$ and $\delta = O(R^{-1/2})$, so that the retained terms are $O(\epsilon R^{3/2})$ and the largest neglected terms are $O(\epsilon^2 R^{3/2})$. Thus the neglected terms are still smaller than those retained by a factor ϵ, and the validity of the approximation is unaffected by the presence of a boundary layer.

4. The forces experienced by the cylinder

The solution obtained in § 2 can be used to calculate the forces exerted on the cylinder by the fluid.

It may be shown that on the surface of the cylinder

$$p_{r\theta} = -\frac{\mu V}{a}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)_{r=1} \tag{37}$$

and

$$p_{rr} = -p(1,\theta), \tag{38}$$

where $p(1,\theta)$ satisfies the equation

$$\frac{\partial p(1,\theta)}{\partial\theta} = -\frac{\mu V}{a}\left(\frac{\partial^3\psi}{\partial r^3} + \frac{\partial^2\psi}{\partial r^2} + 1\right)_{r=1}. \tag{39}$$

If the solution $\psi_0 + \epsilon\psi_1$ found in § 2 is used to calculate these stresses, the force (X, Y) acting on the cylinder can be determined by integration, and one finds that

$$X = 0, \tag{40}$$

$$Y = -2\pi a V\rho U. \tag{41}$$

Thus the drag force is zero to the order of the approximation considered. This is perhaps a surprising result, but it follows from the character of the flow at large distances from the cylinder. There is no wake in this flow, since the boundary layers are closed, and consequently the flow at great distances is (to this order) everywhere irrotational. It follows from momentum considerations (Taylor 1925) that the drag force is zero.

The result (41) for the lifting force is also in agreement with Taylor's conclusions (discussed in § 1), since, as was shown in § 3, the circulation is $2\pi aV$ to the first order.

The above results have been derived from the first approximation $\psi_0 + \epsilon\psi_1$ which, as was shown in § 2, provides a valid approximation to the flow pattern for $R > 0$. If attention is restricted to the case $p > 2$, the second approximation to the flow field can be inserted in the above expressions for the stresses, and the total force on the cylinder recalcuated. One finds that the second approximation makes no contribution to either X or Y, so that

$$X = O(\epsilon^3), \tag{42}$$

$$Y = -2\pi a V\rho U\{1 + O(\epsilon^2)\}. \tag{43}$$

The result (42) is not surprising in view of the argument given above, and (43) is in agreement with the value obtained when the circulation at large distances calculated in § 3 is inserted in Taylor's formula for the lift. Unfortunately the measurements that have been made of the forces experienced by a rotating circular cylinder (Goldstein 1938, pp. 545–548) have been made at high Reynolds numbers where the stability of the flow is in doubt, so that no comparison of the above theory with experiment seems to be possible.

The viscous dissipation in the boundary layer absorbs the work which must be done against the torque to maintain the cylinder's rotation. The torque G can be calculated from (37), and, using the second approximation one finds that

$$G(R) = -4\pi\mu a V\left[1+\frac{\epsilon^2 R^2}{4p(p^2-1)}\right]. \tag{44}$$

For large values of R, p may be replaced by its asymptotic expression, and

$$G \sim -4\pi\mu a V(1+\tfrac{1}{2}\sqrt{2}\epsilon^2 R^{1/2}). \tag{45}$$

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